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Unstructured Adiabatic Quantum Search

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In the adiabatic quantum computation model, a computational procedure is described by the continuous time evolution of a time dependent Hamiltonian. We apply this method to the Grover's problem, i.e., searching a marked item in an unstructured database. Classically, the problem can be solved only in a running time of order $O(N)$ (where N is the number of items in the database), whereas in the quantum model a speed up of order $O(\sqrt{N})$ has been obtained. We show that in the adiabatic quantum model, by a suitable choice of the time-dependent Hamiltonian, it is possible to do the calculation in constant time, independent of the the number of items in the database. However, in this case the initial time-complexity of $O(\sqrt{N})$ is replaced by the complexity of implementing the driving Hamiltonian.

KEY WORDS: quantum computation; adiabatic quantum search.

Recently, a newer subfield emerged by new works addressing the idea of developing quantum algorithms based on adiabatic evolution (Farhi *et al.*) In the adiabatic quantum computation model, a computational procedure is described by the continuous time evolution of a time-dependent Hamiltonian. Here, we apply this method to the Grover problem (i.e., searching a marked item in an unstructured database) (Grover, 1997). Classically, Grover's problem can be solved only in a running time of order $O(N)$ (where N is the number of items in the database), whereas in the standard quantum model a speed up of order $O(\sqrt{N})$ has been obtained (Grover, 1997). The same speed up, of order $O(\sqrt{N})$, has been obtained in the adiabatic evolution model, by using a local formulation of the adiabatic theorem (Roland and Cerf, 2002; van Dam, Mosca, and Vazirani, 2001). Recently, it has been shown that the problem can be solved in constant time, by increasing exponentially the lowest eigenvalue of the system, to a maximum of $\propto \sqrt{N}$, during the computation (Das, Kobes, and Kunstatter, 2003). Here, we show that by making a suitable choice of the time-dependent Hamiltonian, the computation can be done at constant time (independent of the number of items in the database), without increasing exponentially the lowest eigenvalue of the system.

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Let us briefly recall the adiabatic theorem (Brandsen and Joachain, 2000). Consider a quantum system in a state $|\psi(t)\rangle$, which evolves according to the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \tag{1}$$

where $\hat{H}(t)$ is the Hamiltonian of the system (we let $\hbar = 1$). If this Hamiltonian is time independent and the system is initially in its ground state, then it will remain in this state. The adiabatic theorem expresses that if the Hamiltonian varies slowly enough, the state of the system will stay close to the instantaneous ground state of the Hamiltonian at each time t. More specifically, if $|E_0;t\rangle$ and $|E_1;t\rangle$ are the ground and first excited states of the Hamiltonian $\hat{H}(t)$, with energies E_0 and E_1, we define the minimum gap between these eigenvalues

$$\delta_{\min} = \min_{0\le t\le T}[E_1(t) - E_0(t)] \tag{2}$$

and the maximum value of the matrix element of $d\hat{H}(t)/dt$ between the eigenstates as

$$\Delta_{\max} = \max_{0\le t\le T}\left|\left\langle\frac{d\hat{H}}{dt}\right\rangle_{1,0}\right| = \max_{0\le t\le T}\left|\left\langle E_1;t\left|\frac{d\hat{H}}{dt}\right|E_0;t\right\rangle\right| \tag{3}$$

The adiabatic theorem states that if we prepare the system at time $t = 0$ in its ground state $|E_0;t\rangle$ and let it evolve under the Hamiltonian $\hat{H}(t)$ for a time T, then

$$|\langle E_0;T|\psi(T)\rangle|^2 \ge 1 - \varepsilon^2 \tag{4}$$

provided that

$$\Delta_{\max}\delta_{\min}^{-2} \le \varepsilon \tag{5}$$

where $\varepsilon \ll 1$.

This result can be used to design a new type of quantum algorithm based on a time-dependent Hamiltonian (Farhi *et al.*,). Assume, we can build a Hamiltonian for which we know that the ground state encodes the solution of a problem. Then, it suffices to prepare the system in the ground state of another Hamiltonian, easy to build, and change progressively this Hamiltonian into the other one in order to get, after measurement, the sought solution with large probability. The adiabatic theorem imposes the minimum time it takes for this switching to be adiabatic.

Grover's problem can be formulated in the following abstract way (Grover, 1992). Given a Boolean function

$$f(x) = \begin{cases} 1, & x = v \\ 0, & x \ne v \end{cases}, \quad x = 0, 1, \ldots, N-1, \quad v \in \{0, 1, \ldots, N-1\} \tag{6}$$

the problem is to find the value of ν. The database is composed of all the eigenstates of the quantum system, and it is defined as the uniform superposition of these eigenstates ($N = 2^n$, for a quantum computer with n qubits)

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{n-1} |x\rangle \tag{7}$$

On a classical computer, we have to evaluate the function $N/2$ times on average to find the value of ν. In contrast, Grover's quantum algorithm finds ν in $O(\sqrt{N})$ steps (Grover, 1992). Farhi *et al*. have solved Grover's search problem using the adiabatic evolution approach, but this unfortunately resulted in a complexity of order $O(N)$, that is no better than a classical algorithm that checks all possible solutions (Farhi *et al*.,). Roland *et al*. have shown that by dividing T into infinitesimal time intervals dt and applying the adiabaticity condition locally to each of these intervals, one can vary the evolution rate continuously in time, thereby speeding up the computation to $O(\sqrt{N})$ steps (Roland and Cerf, 2002). The same result has been obtained by van Dam *et al*. (van Dam, Mosca, and Vazirani, 2001). Recently, Das *et al*. have shown that the search for a marked item in an unstructured database can be achieved in constant time by increasing exponentially the lowest eigenvalue of the system to a maximum of $\propto \sqrt{N}$ during the computation (Das, Kobes, and Kunstatter, 2003). Also, a similar result has been obtained by Bae *et al*. (Bae and Kwon, 2002, 2003a,b).

The adiabatic quantum algorithms have generally been studied in the case where the "straight line" interpolation from initial ($\hat{H}_0$) to final ($\hat{H}_1$) Hamiltonian is taken

$$\hat{H}(s) = (1 - s)\hat{H}_0 + s\hat{H}_1 \tag{8}$$

where $s = t/T$. Here, $|\psi_0\rangle$ is the ground state of the initial Hamiltonian $\hat{H}_0 = \hat{I} - |\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle = |\nu\rangle$ is the ground state of the final Hamiltonian $\hat{H}_1 = \hat{I} - |\psi_1\rangle|\psi_1\rangle|$ ($\hat{I}$ is the identity operator). Thus the above Hamiltonian reads

$$\hat{H}(s) = \hat{I} - (1 - s)\hat{H}'_0 - s\hat{H}'_1 \tag{9}$$

where,

$$\begin{aligned} \hat{H}'_0 &= |\psi_0\rangle\langle\psi_0| = \hat{I} - \hat{H}_0 \\ \hat{H}'_1 &= |\psi_1\rangle\langle\psi_1| = \hat{I} - \hat{H}_1 \end{aligned} \tag{10}$$

In this work, we are extending this investigation by considering nonlinear interpolation. More exactly, we are considering the following general Hamiltonian

$$\hat{H}(s) = \hat{I} - A(s)\hat{H}'_0 - B(s)\hat{H}'_1 - C(s)\{\hat{H}'_0, \hat{H}'_1\} \tag{11}$$

where $A(s)$, $B(s)$, and $C(s)$ are nonlinear interpolation functions and

$$\{\hat{H}'_0, \hat{H}'_1\} = \hat{H}'_0\hat{H}'_1 + \hat{H}'_1\hat{H}'_0$$

is the anticommutator of $\hat{H}'_0$ and $\hat{H}'_1$.

The problem of simulating $\hat{H}(s)$ with a quantum circuit reduces to alternately simulate the Hamiltonians $\hat{H}'_0$ and $\hat{H}'_1$, for short-time increments Δt. These Hamiltonians are easily simulated using the methods described by Nielsen and Chuang (2000). For example $\hat{H}'_1$ can be implemented via two queries to the database oracle and $\hat{H}'_0$ involves two Hadamard transformations.

There is no reason not to consider nonlinear interpolation. The adiabatic algorithm will work taking any path $\hat{H}(s)$, as long as the adiabaticity condition is satisfied, the ground state at $s = 0$ is $|\psi_0\rangle$ and the ground state at $s = 1$ is $|\psi_1\rangle$. Therefore, $\hat{H}(s)$ must satisfy the following conditions:

$$\begin{aligned}\hat{H}(0) &= \hat{H}_0 = \hat{I} - |\psi_0\rangle\langle\psi_0| \\ \hat{H}(1) &= \hat{H}_1 = \hat{I} - |\psi_1\rangle\langle\psi_1|\end{aligned} \tag{12}$$

In what follows, we will show how one can construct such a Hamiltonian.

The initial state of the search algorithm (7) can be written as

$$|\psi_0\rangle = \sqrt{\frac{N-1}{N}}|\sigma\rangle + \sqrt{\frac{1}{N}}|\nu\rangle \tag{13}$$

where $|\nu\rangle$ and $|\sigma\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq\nu}|x\rangle$ are orthogonal ($\langle\sigma|\nu\rangle = 0$). More generally, we can write $|\psi_0\rangle$ as

$$|\psi_0(\alpha)\rangle = \cos\left(\frac{\pi}{2}\alpha\right)|\sigma\rangle + \sin\left(\frac{\pi}{2}\alpha\right)|\nu\rangle \tag{14}$$

where the parameter $\alpha = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{1}{N}}\right) \in (0, \frac{1}{\sqrt{2}}]$ is a priori known because it involves only the number of qubits in the system. We would like to evolve adiabatically this initial state to the final state

$$|\psi_1(\alpha)\rangle = |\nu\rangle \tag{15}$$

Let us consider the wave function

$$|\psi(\alpha, s)\rangle = \frac{\cos\left(\frac{\pi}{2}s\right)|\psi_0(\alpha)\rangle + \sin\left(\frac{\pi}{2}s\right)|\psi_1(\alpha)\rangle}{\sqrt{1+\sin(\pi S)\sin\left(\frac{\pi}{2}\alpha\right)}} \tag{16}$$

where $s = t/T \in [0, 1]$ is the interpolation parameter. One can see that $|\psi(\alpha, s)\rangle$ performs a nonlinear interpolation from the initial wave function $|\psi_0(\alpha)\rangle$ to the final wave function $|\psi_1(\alpha)\rangle$

$$\begin{aligned}|\psi(\alpha, 0)\rangle &= |\psi_0(\alpha)\rangle \\ |\psi(\alpha, 1)\rangle &= |\psi_1(\alpha)\rangle\end{aligned} \tag{17}$$

Now, let us consider the following wave function:

$$|\varphi(\alpha, s)\rangle = \frac{1}{\cos\left(\frac{\pi}{2}\alpha\right)\sqrt{1+\sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)}} \times \left\{\left[\sin\left(\frac{\pi}{2}s\right)+\cos\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}\alpha\right)\right]|\psi_0(\alpha)\rangle - \left[\cos\left(\frac{\pi}{2}s\right)+\sin\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}\alpha\right)\right]|\psi_1(\alpha)\rangle\right\}. \quad (18)$$

It is easy to show that these two wave functions are orthogonal

$$\langle\psi(\alpha, s)|\varphi(\alpha, s)\rangle = 0$$

Also, one can show that $|\psi(\alpha, s)\rangle$ is the groundstate, with eigenvalue $E_0 = 0$, and $|\psi(\alpha, s)\rangle$ is the first excited state, with eigenvalue $E_1 = 1 - \omega\sin(\pi s)$, of the following time-dependent Hamiltonian

$$\hat{H}(\alpha, s, \omega) = \hat{I} - |\psi(\alpha, s)\rangle\langle\psi(\alpha, s)| - \omega\sin(\pi s)|\psi(\alpha, s)\rangle\langle\varphi(\alpha, s)| \quad (19)$$

where $\hat{I}$ is the identity operator and $\omega \in (0, 1)$ is a free parameter (at our choice). The higher energy eigenvalue is $(N-2)$-fold degenerate, $E_i = 1, i = 2, \ldots, N-1$. Thus, the ground state at $s = 0$ is $|\psi_0(\alpha)\rangle$ and the ground state at $s = 1$ is $|\psi_1(\alpha)\rangle$.

One can easily show that the Hamiltonian (19) is of the form (11)

$$\hat{H}(\alpha, s, \omega) = \hat{I} - A(\alpha, s, \omega)\hat{H}'_0 - B(\alpha, s, \omega)\hat{H}'_1 - C(\alpha, s, \omega)\{\hat{H}'_0, \hat{H}'_1\} \quad (20)$$

where

$$A(\alpha, s, \omega) = \frac{\cos^2\left(\frac{\pi}{2}s\right)\cos^2\left(\frac{\pi}{2}\alpha\right)}{\cos^2\left(\frac{\pi}{2}\alpha\right)\left[1+\sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)\right]} + \frac{\omega\sin(\pi s)\left[\sin\left(\frac{\pi}{2}s\right)+\cos\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}\alpha\right)\right]^2}{\cos^2\left(\frac{\pi}{2}\alpha\right)\left[1+\sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)\right]}$$

$$B(\alpha, s, \omega) = \frac{\sin^2\left(\frac{\pi}{2}s\right)\cos^2\left(\frac{\pi}{2}\alpha\right)}{\cos^2\left(\frac{\pi}{2}\alpha\right)\left[1+\sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)\right]} + \frac{\omega\sin(\pi s)\left[\cos\left(\frac{\pi}{2}s\right)+\sin\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}\alpha\right)\right]^2}{\cos^2\left(\frac{\pi}{2}\alpha\right)\left[1+\sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)\right]}$$

$$C(\alpha, s, \omega) = \frac{\sin(\pi s)}{\cos^2\left(\frac{\pi}{2}\alpha\right)\sin\left(\frac{\pi}{2}\alpha\right)\left[1+\sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)\right]}$$

$$\times \left\{ \frac{1}{2}\cos^2\left(\frac{\pi}{2}\alpha\right) - \omega\left[\sin\left(\frac{\pi}{2}s\right) + \cos\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}\alpha\right)\right] \right.$$
$$\left. \times \left[\cos\left(\frac{\pi}{2}s\right) + \sin\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}\alpha\right)\right]\right\}$$

are the nonlinear interpolation functions we were looking for. Also, one can easily verify that the conditions (12) are satisfied because

$$A(\alpha, 0, \omega) = B(\alpha, 1, \omega) = 1$$
$$A(\alpha, 1, \omega) = B(\alpha, 0, \omega) = 0$$
$$C(\alpha, 0, \omega) = C(\alpha, 1, \omega) = 0$$

The energy gap is independent of α

$$\delta(s, \omega) = 1 - \omega\sin(\pi s) \in (0, 1) \tag{21}$$

Also, taking into account that $\frac{d\hat{H}}{dt} = \frac{ds}{dt}\frac{d\hat{H}}{ds}$, we obtain

$$\Delta(\alpha, t, \omega) = \left|\frac{ds}{dt}\right| \left|\langle\psi(\alpha, s)\left|\frac{d\hat{H}(\alpha, s, \omega)}{ds}\right|\varphi(\alpha, s)\rangle\right| = \frac{1}{T}\Delta(\alpha, s, \omega) \tag{22}$$

The matrix element can be calculated analytically and it is given by

$$\Delta(\alpha, s, \omega) = \frac{\pi}{2}[1 - \omega\sin(\pi s)]\left|\frac{\cos\left(\frac{\pi}{2}\alpha\right)}{1 + \sin(\pi s)\sin\left(\frac{\pi}{2}\alpha\right)}\right| \tag{23}$$

One can easily show that $0 < \Delta(\alpha, s, \omega) \lesssim \frac{\pi}{2}$. Therefore, the extreme values for the gap and the matrix elements are

$$\delta_{\min} = 1 - \omega, \triangle_{\max} = \frac{\pi}{2T} \tag{24}$$

Thus, the minimum running time is given by

$$T(\omega) = \frac{\pi}{2\varepsilon}(1 - \omega)^{-2} \tag{25}$$

For small ω ($\omega \ll 1$) the minimum running time is $T \cong \frac{\pi}{2\varepsilon}$.

In conclusion, we have applied the adiabatic evolution method to Grover's problem, i.e., the search for a marked item in an unstructured database. We have shown that by making a suitable choice of the time-dependent Hamiltonian, the computation can be done in constant time (independent of the number of items in the database), without increasing exponentially the lowest eigenvalue of the system, to a maximum of $\propto \sqrt{N}$, like in Das, Kobes, and Kunstatter (2003). One may think that the $O(1)$ speedup violates the proof that the quadratic speedup is optimal in quantum search (Zalka, 1999). However, it has already been shown that in a continuous time algorithm, based on Hamiltonian evolution, the proof can be

violated (Das, Kobes, and Kunstatter, 2003; Bae and Kwon, 2002, 2003a,b). The adiabatic analogue (Andrecut and Ali, 2004) of the Margoulus and Levitin theorem (Margolus and Levitin, 1998) and the recent argument of Giovannetti, Loyd, and Maccone (2003) support this result. It is worth noticing that in this case, the initial time-complexity of $O(\sqrt{N})$ is replaced by the complexity of implementing the driving Hamiltonian.

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